On the quasi-linear diffusion in collisionless plasmas
(to say nothing about Landau damping)

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General quasi-linear diffusion coefficients for nonrelativistic collisionless plasmas are derived for unstable modes and analytically continued to damped modes. Properties of the resulting diffusion are investigated and discussed.

I. INTRODUCTION

Kinetic linear dispersion in collisionless plasmas for damped waves naturally involves an analytic continuation of the dispersion relation from the unstable region.1 On the nonlinear level an evolution of kinetic waves may be described to some extend in a quasi-linear approach. This approximation assumes a superposition of incoherent linear modes which at the second order (in wave amplitude) affect the background averaged particle distribution functions.2,3 For unstable waves this approximation gives rise to a quasi-linear diffusion; for particles in the electrostatic case one gets (for symbol definition see Appendix A)

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} \]  

(1)

where the diffusion coefficient \( D \) depends on the wave spectrum through Landau resonances with the different modes as

\[ \propto \frac{\gamma}{(\omega_r - kv)^2 + \gamma^2} \]  

(2)

with a resonance broadening due to the finite growth rate. The derivation is of (2) is simple but the question of its validity is difficult; such a diffusion can be an intrinsic property of systems with sufficient overlapping of one-mode resonances/trapped region.4

The Landau resonances in the limit \( |\gamma| \ll |\omega_r| \) become

\[ \propto \pi \delta(\omega_r - kv) \]  

(3)

and the resonance broadening disappears. In this limit one gets the expression for the damping/growth rate as \( \gamma \propto \partial f/\partial v |_{v=\omega_r/k} \) and this approach makes a self-consistent system with energy and linear momentum conservation.5,6

In a general case, for instance for nonpropagating7 or reactive8 instabilities the resonance broadening must be taken into account. In this case for damped modes relation (2) gives negative diffusion coefficients and in the limit \( \gamma \to 0^- \) one gets a negative value of (3), \( \propto -\pi \delta(\omega_r - kv) \), so that there is a discontinuity of the diffusion coefficients at \( \gamma = 0 \). To overcome these problems it is sometimes argued that causality9 requires to take

II. ELECTROSTATIC CASE

A. Unstable case

Let us start with a simple electrostatic case and unstable waves. In this case the dispersion relation for a mode with a wave vector \( k \) and a (complex) frequency \( \omega = \omega_r(k) + i\gamma(k) \) where \( \gamma > 0 \) is given implicitly as

\[ D(k, \omega) = 1 - \sum_s \frac{\omega_s^2}{k} \int_{-\infty}^{\infty} \frac{\partial f_s}{kv - \omega} dv = 0. \]  

(5)

For symbol definition see Appendix A Assuming the quasi-linear approximation, an superposition of incoherent linear waves and slowly varying particle velocity distribution functions \( f_s \)

\[ E_1 = \sum_{\text{modes}} \delta E(k, \omega) e^{ikx - i\omega t} \]  

(6)

\[ f_{s1} = i \frac{q_s}{m_e} \sum_{\text{modes}} \delta E(k, \omega) \frac{e^{ikx - i\omega t}}{kv - \omega} \frac{\partial f_s}{\partial v} \]  

(7)
satisfying the linear dispersion relation (5) for \( f_s \). The linear terms \( E_1 \) and \( f_{s1} \) are assumed to be real and, consequently, \( \omega(-k) = -\omega(k) \) and \( \delta E(-k, -\omega) = \delta E(k, \omega) \); note that \( D(-k, -\omega) = \overline{D(k, \omega)} \).

Taking the second order contribution to the changes of the averaged \( f_s = f_s(v|, v_\perp) \) as

\[
\frac{\partial f_s}{\partial t} = -\frac{q_s}{m_s} \left\langle E_1 \frac{\partial f_{s1}}{\partial v} \right\rangle
\]

where \( \langle \rangle \) denotes space and (fast) time averaging leads to the diffusion equation for the particle distribution functions

\[
\frac{\partial f_s}{\partial t} = \frac{\partial}{\partial v} D_s \frac{\partial f_s}{\partial v}
\]

with

\[
D_s = \frac{q_s^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 3 \frac{1}{kv - \omega}
\]

Using the dispersion relation (5) it is easy to show that the total particle momentum is conserved

\[
\frac{\partial}{\partial t} \sum_s n_s m_s \int_{-\infty}^{\infty} v f_s dv = 0
\]

From (5) also follows that the total (particle and electric) energy is conserved with energy exchanges between particles and waves:

\[
\frac{\partial}{\partial t} \sum_s n_s m_s \int_{-\infty}^{\infty} \frac{v^2}{2} f_s dv = -c_0 \sum_{\text{modes}} \gamma |\delta E(k, \omega)|^2.
\]

Note that in the limit \( \gamma \to 0^+ \) the diffusion coefficient becomes

\[
\lim_{\gamma \to 0^+} D_s = \frac{q_s^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 2 \pi |\delta(\omega - kv)|
\]

and in this limit the total particle kinetic energy and the electrostatic wave energy are conserved

\[
\frac{\partial}{\partial t} \sum_s n_s m_s \int_{-\infty}^{\infty} \frac{v^2}{2} f_s dv = 0
\]

\[
\lim_{\gamma \to 0} \frac{\partial}{\partial t} \sum_{\text{modes}} c_0 |\delta E(k, \omega)|^2 = 0
\]

and there is no energy exchange between particles and waves.

**B. Stable case**

In the stable case \( \gamma < 0 \) the electrostatic dispersion relation may be written in the following form

\[
D = 1 - \sum_s \frac{\omega_{ps}^2}{k} \int_{-\infty}^{\infty} \left[ \frac{1}{kv - \omega} + 2\pi i \delta(kv - \omega) \right] \frac{\partial f_s}{\partial v} dv = 0.
\]

where \( \delta \) is the complex extension of the Dirac \( \delta \) function (see Appendix B). Repeating the quasi-linear procedure with the same form of \( f_{s1} \), Eq. (7) leads to the same quasi-linear diffusion coefficients (10); in this case, however, the diffusion coefficients are negative and the residue terms in the dispersion relation (16) lead to a nonconservation of the linear momentum and energy which makes such a choice unphysical. Similarly taking the absolute value

\[
D_s = \frac{q_s^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 \left| \frac{1}{kv - \omega} + 2\pi i \delta(kv - \omega) \right|
\]

the quasi-linear diffusion does not conserve the linear momentum and energy. Instead, the linear term \( f_{s1} \) has to be chosen as an analytical continuation

\[
\frac{\partial f_s}{\partial t} = i \frac{q_s}{m_s} \sum_{\text{modes}} \delta E(k, \omega) e^{ikx - i\omega t}
\]

Here \( f_{s1} \) is assumed to be real real, and, consequently, \( \omega(-k) = -\omega(k) \) and \( \delta E(-k, -\omega) = \delta E(k, \omega) \); note that \( D(-k, -\omega) = \overline{D(k, \omega)} \) as in the unstable case.

Taking the second order contribution (8) one gets the diffusion coefficient

\[
D_s = \frac{q_s^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 \left[ \frac{1}{kv - \omega} + 2\pi i \delta(kv - \omega) \right]
\]

where the new term contains the complex extension of the Dirac \( \delta \) function which directly corresponds to the residue in the dispersion relation (16). Using the diffusion coefficient (19) one recovers the conservation of the linear momentum (11) and the energy (12).

The diffusion coefficients are positive for \( \gamma > 0 \) as well as in the limit \( \gamma \to 0 \). The analytically continued coefficients must therefore be positive at least for weak damping rates. The analytically continued diffusion coefficients are difficult to understand from the physical point of view. For weak damping rates one can estimate the complex Dirac delta part of the diffusion coefficients using the Taylor expansion of the distribution function as

\[
\frac{\partial}{\partial v} (\pi \delta(kv - \omega)) \frac{\partial f_s}{\partial v} \approx \frac{\partial}{\partial v} \delta(kv - \omega_r) \left( \frac{\partial f_s}{\partial v} - \frac{\gamma^2}{2kv - \omega_r} \right)
\]

and we have to the second order in \( \gamma \)

\[
\frac{\partial f_s}{\partial t} = \frac{\partial}{\partial v} \tilde{D}_s \frac{\partial f_s}{\partial v} + \frac{\partial}{\partial v} \mathcal{H}_s \frac{\partial^3 f_s}{\partial v^3}
\]

where

\[
\tilde{D}_s = \frac{q_s^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 \left[ \frac{1}{kv - \omega} + 2\pi i \delta(kv - \omega_r) \right]
\]

\[
\mathcal{H}_s = -\frac{q_s^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 \frac{\gamma^2}{k_v^2} \delta(kv - \omega_r).
\]
This result indicates that the quasi-linear diffusion for modes with nonnegligible amplitudes and damping rates involves higher order derivatives of the velocity distribution function, i.e., hyper-diffusion in the velocity space.

The presented quasi-linear results are continuous when passing from $\gamma > 0$ to $\gamma < 0$. Taking now the limit $\gamma \to 0$ one recovers the diffusion coefficient (13). A general formulation for the dispersion relation and the diffusion coefficient can be written defining an analytic continuation of $1/x$ in the distributive sense as

$$
\mathcal{E}^1_x = \begin{cases} \frac{1}{x} + 2\pi i \delta(x) & \exists x > 0 \\
\frac{P}{x} + \pi i \delta(x) & \exists x = 0 \\
\frac{1}{x} & \exists x < 0 
\end{cases}
$$

where $P$ denotes principal value (compare with Sokhotski-Plemelj formula). In this way the linear dispersion relation simply reads

$$1 - \sum_s \frac{\omega^2_{ps}}{k} \int_{-\infty}^{\infty} \mathcal{E} \frac{1}{k_v - \omega} \frac{\partial f_s}{\partial v} \, dv = 0 \quad (25)
$$

and the diffusion coefficient is given by

$$D_s = \frac{q^2}{m_s^2} \sum_{\text{modes}} |\delta E(k, \omega)|^2 \Im \left( \frac{1}{k_v - \omega} \right). \quad (26)
$$

### III. ELECTROMAGNETIC CASE

The electrostatic results can be generalized to the general electromagnetic case with the dispersion relation $\det \mathbf{D} = 0$ where the dispersion tensor $\mathbf{D}$ is

$$\mathbf{D} = (k^2c^2 - \omega^2 + \omega^2_p) \mathbf{1} - \mathbf{k} \mathbf{k}^T + \sum_s \omega^2_{ps} \int_{-\infty}^{\infty} \frac{1}{k_v - \omega} \frac{\partial f_s}{\partial v} \, dv \mathbf{v} \mathbf{v}^T \quad (27)
$$

and the tensor $T_{ns}$ may be given as

$$T_{ns} = \left( \begin{array}{cccc}
\frac{n^2 \omega^2}{k^2} J_n^2 & -i \frac{n \omega_{cs}}{k} v_n J_n J_n' & \frac{n \omega_{cs}}{k} v_n J_n'^2 \\
-i \frac{n \omega_{cs}}{k} v_n J_n J_n' & \frac{n^2 \omega^2}{k^2} v_n J_n'^2 & \frac{n \omega_{cs}}{k} v_n J_n J_n' \\
\frac{n \omega_{cs}}{k} v_n J_n J_n' & \frac{n \omega_{cs}}{k} v_n J_n J_n' & \frac{n^2 \omega^2}{k^2} v_n J_n'^2 \\
\end{array} \right) \quad (28)
$$

in the frame where $\mathbf{k} = (k_{\perp0}, k_0)$ and where $J_n$ and $J_n'$ have $\lambda_s = k_{\perp0} v_n / \omega_{cs}$ as the argument.

The quasi-linear approximation assumes a superposition of linear modes

$$E_1 = \sum_{\text{modes}} \delta \mathbf{E}(\mathbf{k}, \omega)e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t} \quad (29)
$$

$$B_1 = \sum_{\text{modes}} \delta \mathbf{B}(\mathbf{k}, \omega)e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t} \quad (30)
$$

where $\delta \mathbf{B}(\mathbf{k}, \omega) = \mathbf{k} \times \delta \mathbf{E}(\mathbf{k}, \omega) / \omega$ and

$$f_{s1} = \frac{i q_s}{m_s} \sum_{\text{modes}} \sum_{n,l=0}^{\infty} \mathcal{E} k_{||} v_n + l \omega_{cs} - \omega \left( \begin{array}{c}
\frac{1}{k_v - \omega} \\
\frac{\partial f_s}{\partial v} \cdot \delta \mathbf{E} \cdot \left( a_{||s} \frac{\partial f_s}{\partial v} + a_{\perp s} \frac{\partial f_s}{\partial v} \cdot \mathbf{J}_n e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t + i(l-n)\varphi} \right) \right) \quad (31)
$$

where the two vectors $a_{||s}$ and $a_{\perp s}$ may be given in the frame where $\mathbf{k} = (k_{\perp0}, k_0)$ as

$$a_{||s} = \left( \frac{\int \omega_{cs} k_{\perp0} J_n - i k_{||0} v_n [1 - \omega_{cs} / \omega] J_n}{k_{||0} v_n} \right)
$$

$$a_{\perp s} = \left( \frac{\int \omega_{cs} k_{\perp0} J_n - i k_{||0} v_n [1 - \omega_{cs} / \omega] J_n'}{k_{||0} v_n} \right)
$$

The different modes are assumed to satisfy the linear dispersion relation

$$\det \mathbf{D}(\mathbf{k}, \omega) = 0 \quad \text{and} \quad \mathbf{D}(\mathbf{k}, \omega) \cdot \delta \mathbf{E}(\mathbf{k}, \omega) = 0 \quad (32)
$$

The linear terms $\mathbf{E}_1$, $\mathbf{B}_1$, and $f_{s1}$ are assumed to be real and, consequently, $\omega(-\mathbf{k}) = -\omega(\mathbf{k})$, $\delta \mathbf{E}(-\mathbf{k}, -\omega) = \delta \mathbf{E}(\mathbf{k}, \omega)$, and $\delta \mathbf{B}(-\mathbf{k}, -\omega) = -\delta \mathbf{B}(\mathbf{k}, \omega)$.

Taking the second order contribution to the changes of the averaged $f_s = f_s(v_n, v_{\perp})$ as

$$\frac{\partial f_s}{\partial t} = -\frac{q_s}{m_s} \left( \langle \mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1 \rangle \cdot \frac{\partial f_{s1}}{\partial \mathbf{v}} \right) \quad (33)
$$

where $\langle \rangle$ denotes space and (fast) time average leads to the diffusion equation for the particle distribution functions

$$\frac{\partial f_s}{\partial t} = \frac{\partial}{\partial v} \left( D_{||s} \frac{\partial f_s}{\partial v} + D_{\perp s} \frac{\partial f_s}{\partial v_{\perp}} \right) + \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} \left( D_{||s} \frac{\partial f_s}{\partial v} + D_{\perp s} \frac{\partial f_s}{\partial v_{\perp}} \right) \quad (34)
$$

The diffusion coefficients may be given in the following explicit form

$$D_{IJ,ns} = \frac{q^2}{m^2} \sum_{\text{modes}} \sum_{n=-\infty}^{\infty} (\delta \mathbf{E} \cdot a_{IJns}) (a_{IJns} \cdot \delta \mathbf{E}) \quad (35)
$$

$$\times \frac{1}{k_{||} v_n + n \omega_{cs} - \omega} \quad (36)
$$

where $I, J = ||, \perp$. For weak damping rates one gets hyper-diffusion terms analogously to the electrostatic case (21).

Using the dispersion relation (27) it is easy to show that the total parallel momentum and energy are conserved

$$\sum_s \frac{\partial p_{cs||}}{\partial t} = -2\epsilon_0 \sum_s \gamma \Re \frac{k_{||} |\delta \mathbf{E}|^2 - k \cdot \delta \mathbf{E} \delta \mathbf{E}|}{\omega} \frac{\partial p_{cs||}}{\partial t} \quad (36)
$$

$$\sum_s \frac{\partial E_{cs||}}{\partial t} = -\sum_s \gamma \left( \epsilon_0 |\delta \mathbf{E}|^2 + \frac{1}{\mu_0} |\delta \mathbf{B}|^2 \right) \frac{\partial E_{cs||}}{\partial t} \quad (37)
$$
In the limit $\gamma \to 0$ there is again no exchange of parallel linear momentum and energy between particles and waves as in the electrostatic case.

**IV. DISCUSSION**

In this paper we investigated quasi-linear properties of the proper kinetic modes in collisionless nonrelativistic plasma; the evolution of transient modes due to the initial condition is beyond the scope of the paper. We show that the general quasi-linear diffusion coefficients as a resonant broadening due to the finite growth rate which are easily derived for unstable modes; need to be analytically continued to negative growth rates in a manner similar to the Landau procedure. We derived the general quasi-linear diffusion coefficients starting in the simple electrostatic case and generalized these results to nonrelativistic magnetized plasma. The derived quasi-linear diffusion conserves the energy and the linear momentum and the exchanges of energy and momentum between fields and particles cease as the growth/damping rate approach zero. The analytic continuation of the diffusion coefficients brings the problem of the physical interpretation similar to the case of the original linear result. Our results indicate that the quasi-linear diffusion of damped waves includes some hyper-diffusion in the velocity space (i.e., terms proportional to higher order derivatives of the particle velocity distribution function).

The limiting forms of the quasi-linear diffusion coefficients as $\gamma \to 0$ are convenient for theoretical and numerical purposes. However, in this limit the energy and momentum exchanges between waves and particles approach zero and consequently the particle diffusion is likely reduced. This is consistent with the reduction of the resonance broadening due to the finite damping/growth rate. Conservation laws naturally constrain the particle diffusion and must be taken into account when analysing particle heating and acceleration by electromagnetic fluctuations.

**Appendix A: Glossary**

Here $t$ denotes the time, and the subscript $s$ denotes different species. Here $i$ denotes the imaginary unit, $\Re$ and $\Im$ denote the real and imaginary part, respectively, and the overline denotes the complex conjugate, $a + ib = a - ib$ for real $a$ and $b$. Here $\omega$ denotes the complex frequency, $\omega_r = \Re \omega$, $\gamma = 3\omega$.

In the electrostatic case $x$ denotes the only spatial variable, $f_\perp$ denotes the ambient (averaged) normalized velocity distribution function, $f_\perp = f_\perp(v)$ where $v$ is the only velocity component; $E_\perp$ denotes the first-order, linear electric field fluctuations, $\delta E$ denote the linear fluctuations of a given mode and $\delta E \perp$ denotes the first-order, linear fluctuations of the velocity distribution function. Here $k$ denotes the only component of the wave vector.

In the electromagnetic case $x$ denotes the position, $E$ and $B$ denote the electric and magnetic fields, respectively, $B_0$ denotes the ambient magnetic field and $B_0 = |B_0|$ denotes its magnitude. Here $f_\perp = f_\perp(v_\parallel, v_\perp)$ denotes the ambient (averaged) normalized velocity distribution function where $v_\parallel$ and $v_\perp$ denote magnitude of the velocity components parallel and perpendicular to $B_0$, respectively. Here $E_\perp$ and $B_\perp$ denote the first-order, linear fluctuations of the electric and magnetic fields, respectively, and $\delta E$ and $\delta B$ denote the linear electric and magnetic fields of a given linear mode. Here $f_\perp = f_\perp(v_\parallel, v_\perp, \varphi)$ the first-order, linear fluctuations of the velocity distribution function where $\varphi$ denotes the gyrophase. Here $k$ denotes the wave vector, and $k_\parallel$ and $k_\perp$ denotes its parallel and perpendicular components with respect to $B_0$, respectively.

Here $\omega_{ce} = q_pB_0/m_s$ and $\omega_{pe} = (n_eq_s^2/m_s\epsilon_0)^{1/2}$ denote the cyclotron and plasma frequencies, respectively. In these expressions $m_s$, $q_s$, and $n_e$ denote the mass, the charge, and the number density, respectively, and $\epsilon_0$ and $\mu_0$ denote the vacuum electric permittivity and magnetic permeability, respectively.

In the electromagnetic case $p_{\parallel}$ stands for the particle parallel linear momentum $p_{\parallel} = m_s n_s \int_{B_0} v_\parallel f_\perp d^3v$ and $\mathcal{E}_s$ denotes the particle kinetic energy $\mathcal{E}_s = m_s n_s \int_{B_0} (v^2/2) f_\perp d^3v$ while $\epsilon_{c_{\parallel}}$ stands for the (averaged) linear momentum of the electromagnetic field $\epsilon_{c_{\parallel}} = \epsilon_0 (E \cdot B_0)/B_0$ and $\mathcal{E}_{c_{\parallel}}$ denotes the (averaged) electromagnetic energy $\mathcal{E}_{c_{\parallel}} = \langle \epsilon_0 |E|^2 + |B|^2/\mu_0 \rangle/2$ where $\langle \rangle$ denote averaging.

**Appendix B: Complex Dirac $\delta$ function**

Complex extension of the Dirac $\delta$ function for a sufficiently smooth (analytic) real function $f$ is defined as

$$\int_a^b \delta(x - c) f(x) dx \equiv \left\{ \begin{array}{ll} f(c) & \text{if } a \leq \Re c \leq b \\ 0 & \text{else.} \end{array} \right. \quad (B1)$$

for real $a, b, x \in \mathbb{R}$ and general complex $c \in \mathbb{C}$. This definition requires that there exists a (unique) analytic continuation of $f$ to $c$.

The complex extension $\delta$ has the following properties for analytical real functions

$$\delta(c) = \overline{\delta(c)} \quad (B2)$$

which follows from the Taylor expansion around $c_r = \Re c$ in $c_i = \Im c$

$$f(c) = \sum_{n=0}^\infty \frac{f^{(n)}(c_r)(ic_i)^n}{n!} = f(c) \quad (B3)$$

where $f^{(n)} = d^n f/dx^n$. 
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